Problem A.2

Consider the collection of all polynomials (with complex coefficients) of degree $< N$ in x.

(a) Does this set constitute a vector space (with the polynomials as "vectors")? If so, suggest a convenient basis, and give the dimension of the space. If not, which of the defining properties does it lack?

(b) What if we require that the polynomials be *even* functions?

(c) What if we require that the leading coefficient (i.e. the number multiplying x^{N-1}) be 1?

(d) What if we require that the polynomials have the value 0 at $x = 1$?

(e) What if we require that the polynomials have the value 1 at $x = 0$?

Solution

In order for a collection of vectors $\mathcal{V}$ to be a vector space over the complex numbers $\mathbb{C}$, the vector addition and scalar multiplication operations defined on it must satisfy the following ten properties.

(A1) $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A2) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \mathcal{V}$.

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathbb{C}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for every $\alpha \in \mathbb{C}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\alpha, \beta \in \mathbb{C}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

Part (a)

Here $\mathcal{V}$ consists of all the polynomials of degree less than N. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let α and β be complex scalars.

$$\mathbf{x} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}$$

$$\mathbf{y} = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$$

$$\mathbf{z} = c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1} x^{N-1}$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) \\ &= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots (a_{N-1}+b_{N-1})x^{N-1}\end{aligned}$$

Because $\mathbf{x}+\mathbf{y}$ is also a polynomial of degree $N-1$, $\mathbf{x}+\mathbf{y}$ is a vector in $\mathcal{V}$. Property A1 is satisfied.

Property A2

Compare the vector sums of $(\mathbf{x}+\mathbf{y})+\mathbf{z}$ and $\mathbf{x}+(\mathbf{y}+\mathbf{z})$.

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= \left[\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right)\right] \\ &\qquad + \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}\right) \\ &= \left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots (a_{N-1}+b_{N-1})x^{N-1}\right] \\ &\qquad + \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}\right) \\ &= [(a_0+b_0)+c_0] + [(a_1+b_1)+c_1]x + [(a_2+b_2)+c_2]x^2 + \cdots + [(a_{N-1}+b_{N-1})+c_{N-1}]x^{N-1} \\ &= (a_0+b_0+c_0) + (a_1+b_1+c_1)x + (a_2+b_2+c_2)x^2 + \cdots + (a_{N-1}+b_{N-1}+c_{N-1})x^{N-1}\end{aligned}$$

$$\begin{aligned}\mathbf{x}+(\mathbf{y}+\mathbf{z}) &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\ &\qquad + \left[\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) + \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}\right)\right] \\ &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\ &\qquad + \left[(b_0+c_0) + (b_1+c_1)x + (b_2+c_2)x^2 + \cdots + (b_{N-1}+c_{N-1})x^{N-1}\right] \\ &= [a_0+(b_0+c_0)] + [a_1+(b_1+c_1)]x + [a_2+(b_2+c_2)]x^2 + \cdots + [a_{N-1}+(b_{N-1}+c_{N-1})]x^{N-1} \\ &= (a_0+b_0+c_0) + (a_1+b_1+c_1)x + (a_2+b_2+c_2)x^2 + \cdots + (a_{N-1}+b_{N-1}+c_{N-1})x^{N-1}\end{aligned}$$

Because of the associative property of addition, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$. Property A2 is satisfied.

Property A3

Compare the vector sums of $\mathbf{x}+\mathbf{y}$ and $\mathbf{y}+\mathbf{x}$.

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) \\ &= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_{N-1}+b_{N-1})x^{N-1} \\ \mathbf{y}+\mathbf{x} &= \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) + \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\ &= (b_0+a_0) + (b_1+a_1)x + (b_2+a_2)x^2 + \cdots + (b_{N-1}+a_{N-1})x^{N-1} \\ &= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_{N-1}+b_{N-1})x^{N-1}\end{aligned}$$

Because of the commutative property of addition, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$. Property A3 is satisfied.

Property A4

The zero element is the vector in $\mathcal{V}$ with all coefficients set to zero.

$$\mathbf{0} = 0 + 0x + 0x^2 + \cdots 0x^{N-1}$$

Adding this to $\mathbf{x}$ results in $\mathbf{x}$.

$$\begin{aligned}\mathbf{x} + \mathbf{0} &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}\right) + \left(0 + 0x + 0x^2 + \cdots 0x^{N-1}\right) \\ &= (a_0 + 0) + (a_1 + 0)x + (a_2 + 0)x^2 + \cdots + (a_{N-1} + 0)x^{N-1} \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} \\ &= \mathbf{x}\end{aligned}$$

Property A4 is satisfied.

Property A5

The additive inverse of $\mathbf{x}$ is
$-\mathbf{x} = -\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}\right) = -a_0 - a_1 x - a_2 x^2 - \cdots - a_{N-1} x^{N-1}$. Because $-\mathbf{x}$ is a polynomial of degree $N-1$, $-\mathbf{x}$ is also in $\mathcal{V}$.

$$\begin{aligned}\mathbf{x} + (-\mathbf{x}) &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}\right) + \left(-a_0 - a_1 x - a_2 x^2 - \cdots - a_{N-1} x^{N-1}\right) \\ &= [a_0 + (-a_0)] + [a_1 + (-a_1)]x + [a_2 + (-a_2)]x^2 + \cdots + [a_{N-1} + (-a_{N-1})]x^{N-1} \\ &= 0 + 0x + 0x^2 + \cdots 0x^{N-1} \\ &= \mathbf{0}\end{aligned}$$

Property A5 is satisfied.

Property M1

Multiply α and $\mathbf{x}$.

$$\begin{aligned}\alpha\mathbf{x} &= \alpha\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}\right) \\ &= \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_{N-1} x^{N-1}\end{aligned}$$

Because $\alpha\mathbf{x}$ is also a polynomial of degree $N-1$, $\alpha\mathbf{x}$ is in $\mathcal{V}$. Property M1 is satisfied.

Property M2

Compare the formulas of $(\alpha\beta)\mathbf{x}$ and $\alpha(\beta\mathbf{x})$.

$$\begin{aligned}
(\alpha\beta)\mathbf{x} &= (\alpha\beta)\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\
&= [(\alpha\beta)a_0] + [(\alpha\beta)a_1]x + [(\alpha\beta)a_2]x^2 + \cdots [(\alpha\beta)a_{N-1}]x^{N-1} \\
&= \alpha\beta a_0 + \alpha\beta a_1 x + \alpha\beta a_2 x^2 + \cdots \alpha\beta a_{N-1}x^{N-1} \\
\alpha(\beta\mathbf{x}) &= \alpha\left[\beta\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right)\right] \\
&= \alpha\left(\beta a_0 + \beta a_1 x + \beta a_2 x^2 + \cdots + \beta a_{N-1}x^{N-1}\right) \\
&= [\alpha(\beta a_0)] + [\alpha(\beta a_1)]x + [\alpha(\beta a_2)]x^2 + \cdots + [\alpha(\beta a_{N-1})]x^{N-1} \\
&= \alpha\beta a_0 + \alpha\beta a_1 x + \alpha\beta a_2 x^2 + \cdots \alpha\beta a_{N-1}x^{N-1}
\end{aligned}$$

Because of the associative property of multiplication, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$. Property M2 is satisfied.

Property M3

Compare the formulas of $\alpha(\mathbf{x}+\mathbf{y})$ and $\alpha\mathbf{x} + \alpha\mathbf{y}$.

$$\begin{aligned}
\alpha(\mathbf{x}+\mathbf{y}) &= \alpha\left[\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right)\right] \\
&= \alpha\left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots (a_{N-1}+b_{N-1})x^{N-1}\right] \\
&= \alpha(a_0+b_0) + \alpha(a_1+b_1)x + \alpha(a_2+b_2)x^2 + \cdots \alpha(a_{N-1}+b_{N-1})x^{N-1} \\
\alpha\mathbf{x} + \alpha\mathbf{y} &= \alpha\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \alpha\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) \\
&= \left(\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_{N-1}x^{N-1}\right) + \left(\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2 + \cdots + \alpha b_{N-1}x^{N-1}\right) \\
&= (\alpha a_0 + \alpha b_0) + (\alpha a_1 + \alpha b_1)x + (\alpha a_2 + \alpha b_2)x^2 + \cdots + (\alpha a_{N-1} + \alpha b_{N-1})x^{N-1}
\end{aligned}$$

Because of the distributive property, $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$. Property M3 is satisfied.

Property M4

Compare the formulas of $(\alpha+\beta)\mathbf{x}$ and $\alpha\mathbf{x} + \beta\mathbf{x}$.

$$\begin{aligned}
(\alpha+\beta)\mathbf{x} &= (\alpha+\beta)\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\
&= [(\alpha+\beta)a_0] + [(\alpha+\beta)a_1]x + [(\alpha+\beta)a_2]x^2 + \cdots + [(\alpha+\beta)a_{N-1}]x^{N-1} \\
&= (\alpha a_0 + \beta a_0) + (\alpha a_1 + \beta a_1)x + (\alpha a_2 + \beta a_2)x^2 + \cdots + (\alpha a_{N-1} + \beta a_{N-1})x^{N-1} \\
\alpha\mathbf{x} + \beta\mathbf{x} &= \alpha\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \beta\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\
&= \left(\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_{N-1}x^{N-1}\right) + \left(\beta a_0 + \beta a_1 x + \beta a_2 x^2 + \cdots + \beta a_{N-1}x^{N-1}\right) \\
&= (\alpha a_0 + \beta a_0) + (\alpha a_1 + \beta a_1)x + (\alpha a_2 + \beta a_2)x^2 + \cdots + (\alpha a_{N-1} + \beta a_{N-1})x^{N-1}
\end{aligned}$$

Because of the distributive property, $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$. Property M4 is satisfied.

Property M5

Multiply 1 and $\mathbf{x}$.

$$\begin{aligned} 1\mathbf{x} &= 1\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) \\ &= a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1} \\ &= \mathbf{x} \end{aligned}$$

Property M5 is satisfied. All ten properties are satisfied, so the set of polynomials under degree N is a vector space over the complex numbers. The dimension of the vector space is N because each of the vectors is spanned by the N basis vectors, 1, x, x^2, …, x^{N-1}.

Part (b)

Here $\mathcal{V}$ is the set of polynomials under degree N that are even functions. The same argument from part (a) applies but with the coefficients that have odd subscripts set to zero. All ten properties are satisfied again, so the set of even polynomials under degree N is a vector space over the complex numbers. If N is even, then the dimension of the vector space is $N/2$ because each of the vectors is spanned by the $N/2$ basis vectors, 1, x^2, x^4, …, x^{N-2}. And if N is odd, then the dimension of the vector space is $(N-1)/2+1$ because each of the vectors is spanned by the $(N-1)/2+1$ basis vectors, 1, x^2, x^4, …, $x^{(N-1)/2}$.

Part (c)

Here $\mathcal{V}$ is the set of polynomials under degree N that have a leading coefficient of 1. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let α and β be complex scalars.

$$\mathbf{x} = a_0 + a_1x + a_2x^2 + \cdots + 1x^{N-1}$$

$$\mathbf{y} = b_0 + b_1x + b_2x^2 + \cdots + 1x^{N-1}$$

$$\mathbf{z} = c_0 + c_1x + c_2x^2 + \cdots + 1x^{N-1}$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= \left(a_0 + a_1x + a_2x^2 + \cdots + 1x^{N-1}\right) + \left(b_0 + b_1x + b_2x^2 + \cdots + 1x^{N-1}\right) \\ &= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + 2x^{N-1} \end{aligned}$$

Although $\mathbf{x}+\mathbf{y}$ is a polynomial of degree $N-1$, the leading coefficient is not 1, which means $\mathbf{x}+\mathbf{y}$ is not a vector in $\mathcal{V}$. Property A1 is not satisfied, so the set of polynomials under degree N that have a leading coefficient of 1 is not a vector space over the complex numbers.

Part (d)

Here $\mathcal{V}$ is the set of polynomials under degree N that are zero at $x = 1$. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let α and β be complex scalars.

$$\mathbf{x} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}; \qquad a_0 + a_1 + a_2 + \cdots + a_{N-1} = 0$$

$$\mathbf{y} = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}; \qquad b_0 + b_1 + b_2 + \cdots + b_{N-1} = 0$$

$$\mathbf{z} = c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}; \qquad c_0 + c_1 + c_2 + \cdots + c_{N-1} = 0$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_{N-1} + b_{N-1})x^{N-1}\end{aligned}$$

So $\mathbf{x} + \mathbf{y}$ is also a polynomial with degree under N. Evaluate it at $x = 1$.

$$\begin{aligned}(\mathbf{x} + \mathbf{y})(1) &= (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_{N-1} + b_{N-1}) \\ &= (a_0 + a_1 + a_2 + \cdots + a_{N-1}) + (b_0 + b_1 + b_2 + \cdots + b_{N-1}) \\ &= 0\end{aligned}$$

Because $\mathbf{x} + \mathbf{y}$ is zero at $x = 1$, $\mathbf{x} + \mathbf{y}$ is a vector in $\mathcal{V}$. Property A1 is satisfied.

Property A2

Compare the vector sums of $(\mathbf{x} + \mathbf{y}) + \mathbf{z}$ and $\mathbf{x} + (\mathbf{y} + \mathbf{z})$.

$$\begin{aligned}(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= \left[\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right)\right] \\ &\qquad + \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}\right) \\ &= \left[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots (a_{N-1} + b_{N-1})x^{N-1}\right] \\ &\qquad + \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}\right) \\ &= [(a_0 + b_0) + c_0] + [(a_1 + b_1) + c_1]x + [(a_2 + b_2) + c_2]x^2 + \cdots + [(a_{N-1} + b_{N-1}) + c_{N-1}]x^{N-1} \\ &= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + (a_2 + b_2 + c_2)x^2 + \cdots + (a_{N-1} + b_{N-1} + c_{N-1})x^{N-1}\end{aligned}$$

$$\begin{aligned}\mathbf{x} + (\mathbf{y} + \mathbf{z}) &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\ &\qquad + \left[\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1}x^{N-1}\right) + \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}\right)\right] \\ &= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1}x^{N-1}\right) \\ &\qquad + \left[(b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 + \cdots + (b_{N-1} + c_{N-1})x^{N-1}\right] \\ &= [a_0 + (b_0 + c_0)] + [a_1 + (b_1 + c_1)]x + [a_2 + (b_2 + c_2)]x^2 + \cdots + [a_{N-1} + (b_{N-1} + c_{N-1})]x^{N-1} \\ &= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + (a_2 + b_2 + c_2)x^2 + \cdots + (a_{N-1} + b_{N-1} + c_{N-1})x^{N-1}\end{aligned}$$

Because of the associative property of addition, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$. Property A2 is satisfied.

Property A3

Compare the vector sums of $\mathbf{x}+\mathbf{y}$ and $\mathbf{y}+\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}+\mathbf{y} &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1x + b_2x^2 + \cdots + b_{N-1}x^{N-1}\right) \\
&= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_{N-1}+b_{N-1})x^{N-1} \\
\mathbf{y}+\mathbf{x} &= \left(b_0 + b_1x + b_2x^2 + \cdots + b_{N-1}x^{N-1}\right) + \left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) \\
&= (b_0+a_0) + (b_1+a_1)x + (b_2+a_2)x^2 + \cdots + (b_{N-1}+a_{N-1})x^{N-1} \\
&= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_{N-1}+b_{N-1})x^{N-1}
\end{aligned}$$

Because of the commutative property of addition, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$. Property A3 is satisfied.

Property A4

The zero element is the vector in $\mathcal{V}$ with all coefficients set to zero.

$$\mathbf{0} = 0 + 0x + 0x^2 + \cdots 0x^{N-1}$$

Adding this to $\mathbf{x}$ results in $\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}+\mathbf{0} &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(0 + 0x + 0x^2 + \cdots 0x^{N-1}\right) \\
&= (a_0+0) + (a_1+0)x + (a_2+0)x^2 + \cdots + (a_{N-1}+0)x^{N-1} \\
&= a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1} \\
&= \mathbf{x}
\end{aligned}$$

Property A4 is satisfied.

Property A5

The additive inverse of $\mathbf{x}$ is
$-\mathbf{x} = -\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) = -a_0 - a_1x - a_2x^2 - \cdots - a_{N-1}x^{N-1}$. Because $-\mathbf{x}$ is a polynomial of degree $N-1$ and is zero at $x=1$, $-\mathbf{x}$ is also in $\mathcal{V}$.

$$\begin{aligned}
\mathbf{x}+(-\mathbf{x}) &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(-a_0 - a_1x - a_2x^2 - \cdots - a_{N-1}x^{N-1}\right) \\
&= [a_0+(-a_0)] + [a_1+(-a_1)]x + [a_2+(-a_2)]x^2 + \cdots + [a_{N-1}+(-a_{N-1})]x^{N-1} \\
&= 0 + 0x + 0x^2 + \cdots 0x^{N-1} \\
&= \mathbf{0}
\end{aligned}$$

Property A5 is satisfied.

Property M1

Multiply α and $\mathbf{x}$.

$$\begin{aligned}\alpha\mathbf{x} &= \alpha\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right)\\ &= \alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \cdots + \alpha a_{N-1}x^{N-1}\end{aligned}$$

Because $\alpha\mathbf{x}$ is also a polynomial of degree $N-1$ and is zero at $x = 1$, $\alpha\mathbf{x}$ is in $\mathcal{V}$. Property M1 is satisfied.

Property M2

Compare the formulas of $(\alpha\beta)\mathbf{x}$ and $\alpha(\beta\mathbf{x})$.

$$\begin{aligned}(\alpha\beta)\mathbf{x} &= (\alpha\beta)\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right)\\ &= [(\alpha\beta)a_0] + [(\alpha\beta)a_1]x + [(\alpha\beta)a_2]x^2 + \cdots [(\alpha\beta)a_{N-1}]x^{N-1}\\ &= \alpha\beta a_0 + \alpha\beta a_1x + \alpha\beta a_2x^2 + \cdots \alpha\beta a_{N-1}x^{N-1}\\ \alpha(\beta\mathbf{x}) &= \alpha\left[\beta\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right)\right]\\ &= \alpha\left(\beta a_0 + \beta a_1x + \beta a_2x^2 + \cdots + \beta a_{N-1}x^{N-1}\right)\\ &= [\alpha(\beta a_0)] + [\alpha(\beta a_1)]x + [\alpha(\beta a_2)]x^2 + \cdots + [\alpha(\beta a_{N-1})]x^{N-1}\\ &= \alpha\beta a_0 + \alpha\beta a_1x + \alpha\beta a_2x^2 + \cdots \alpha\beta a_{N-1}x^{N-1}\end{aligned}$$

Because of the associative property of multiplication, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$. Property M2 is satisfied.

Property M3

Compare the formulas of $\alpha(\mathbf{x}+\mathbf{y})$ and $\alpha\mathbf{x} + \alpha\mathbf{y}$.

$$\begin{aligned}\alpha(\mathbf{x}+\mathbf{y}) &= \alpha\left[\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1x + b_2x^2 + \cdots + b_{N-1}x^{N-1}\right)\right]\\ &= \alpha\left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots (a_{N-1}+b_{N-1})x^{N-1}\right]\\ &= \alpha(a_0+b_0) + \alpha(a_1+b_1)x + \alpha(a_2+b_2)x^2 + \cdots \alpha(a_{N-1}+b_{N-1})x^{N-1}\\ \alpha\mathbf{x} + \alpha\mathbf{y} &= \alpha\left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) + \alpha\left(b_0 + b_1x + b_2x^2 + \cdots + b_{N-1}x^{N-1}\right)\\ &= \left(\alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \cdots + \alpha a_{N-1}x^{N-1}\right) + \left(\alpha b_0 + \alpha b_1x + \alpha b_2x^2 + \cdots + \alpha b_{N-1}x^{N-1}\right)\\ &= (\alpha a_0 + \alpha b_0) + (\alpha a_1 + \alpha b_1)x + (\alpha a_2 + \alpha b_2)x^2 + \cdots + (\alpha a_{N-1} + \alpha b_{N-1})x^{N-1}\end{aligned}$$

Because of the distributive property, $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$. Property M3 is satisfied.

Property M4

Compare the formulas of $(\alpha+\beta)\mathbf{x}$ and $\alpha\mathbf{x}+\beta\mathbf{x}$.

$$\begin{aligned}(\alpha+\beta)\mathbf{x} &= (\alpha+\beta)\left(a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}\right)\\ &= [(\alpha+\beta)a_0]+[(\alpha+\beta)a_1]x+[(\alpha+\beta)a_2]x^2+\cdots+[(\alpha+\beta)a_{N-1}]x^{N-1}\\ &= (\alpha a_0+\beta a_0)+(\alpha a_1+\beta a_1)x+(\alpha a_2+\beta a_2)x^2+\cdots+(\alpha a_{N-1}+\beta a_{N-1})x^{N-1}\\ \alpha\mathbf{x}+\beta\mathbf{x} &= \alpha\left(a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}\right)+\beta\left(a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}\right)\\ &= \left(\alpha a_0+\alpha a_1x+\alpha a_2x^2+\cdots+\alpha a_{N-1}x^{N-1}\right)+\left(\beta a_0+\beta a_1x+\beta a_2x^2+\cdots+\beta a_{N-1}x^{N-1}\right)\\ &= (\alpha a_0+\beta a_0)+(\alpha a_1+\beta a_1)x+(\alpha a_2+\beta a_2)x^2+\cdots+(\alpha a_{N-1}+\beta a_{N-1})x^{N-1}\end{aligned}$$

Because of the distributive property, $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$. Property M4 is satisfied.

Property M5

Multiply 1 and $\mathbf{x}$.

$$\begin{aligned}1\mathbf{x} &= 1\left(a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}\right)\\ &= a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}\\ &= \mathbf{x}\end{aligned}$$

Property M5 is satisfied. All ten properties are satisfied, so the set of polynomials under degree N that are zero at $x=1$ is a vector space over the complex numbers. Before saying what its dimension is, it's necessary to rewrite the $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ vectors as

$$\begin{aligned}\mathbf{x} &= a'_1(x-1)+a'_2(x-1)^2+\cdots+a'_{N-1}(x-1)^{N-1}\\ \mathbf{y} &= b'_1(x-1)+b'_2(x-1)^2+\cdots+b'_{N-1}(x-1)^{N-1}\\ \mathbf{z} &= c'_1(x-1)+c'_2(x-1)^2+\cdots+c'_{N-1}(x-1)^{N-1}.\end{aligned}$$

Doing so combines the conditions that they are polynomials under degree N and that they are zero at $x=1$. The dimension of the vector space is $N-1$ because each of the vectors is spanned by the $N-1$ basis vectors, $x-1$, $(x-1)^2$, …, $(x-1)^{N-1}$.

Part (e)

Here $\mathcal{V}$ is the set of polynomials under degree N that evaluate to 1 at $x=0$. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let α and β be complex scalars.

$$\begin{aligned}\mathbf{x} &= a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}; & a_0 &= 1\\ \mathbf{y} &= b_0+b_1x+b_2x^2+\cdots+b_{N-1}x^{N-1}; & b_0 &= 1\\ \mathbf{z} &= c_0+c_1x+c_2x^2+\cdots+c_{N-1}x^{N-1}; & c_0 &= 1\end{aligned}$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= \left(a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}\right) + \left(b_0 + b_1x + b_2x^2 + \cdots + b_{N-1}x^{N-1}\right) \\ &= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_{N-1}+b_{N-1})x^{N-1}\end{aligned}$$

So $\mathbf{x}+\mathbf{y}$ is also a polynomial with degree under N. Evaluate it at $x = 0$.

$$\begin{aligned}(\mathbf{x}+\mathbf{y})(0) &= (a_0+b_0) \\ &= 1+1 \\ &= 2\end{aligned}$$

Because $\mathbf{x}+\mathbf{y}$ is not 1 at $x = 0$, $\mathbf{x}+\mathbf{y}$ is not a vector in $\mathcal{V}$. Property A1 is not satisfied, so the set of polynomials under degree N that evaluate to 1 at $x = 0$ is not a vector space over the complex numbers.